ON VAN KAMPEN-FLORES, CONWAY-GORDON-SACHS AND RADON THEOREMS

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ABSTRACT. We exhibit relations between van Kampen-Flores, Conway-Gordon-Sachs and Radon theorems, by presenting direct proofs of some implications between them. The key idea is an interesting relation between the van Kampen and the Conway-Gordon-Sachs numbers for restrictions of a map of $(d+2)$-simplex to $\mathbb{R}^d$ to the $(d+1)$-face and to the $[d/2]$-skeleton.

We exhibit relations between the following van Kampen-Flores, Conway-Gordon-Sachs and Radon theorems, by presenting direct proofs of some implications between them, see Main Remark 1 below. Thus we obtain alternative proofs of some of these results assuming another. Direct proofs of the implications (1^+) below were apparently not published before. Such proofs are based on interesting properties of the van Kampen and the Conway-Gordon-Sachs numbers, see Lemma 2 below.

Denote by Δ_N the N-dimensional simplex.

Consider the following assertions for each integer $d>0$:

(VKF_d) *Van Kampen-Flores Theorem.* Let $f:\Delta_{d+2}\to\mathbb{R}^d$ be a general position PL map.

If d is even, then there are disjoint $(d/2)$-faces whose images intersect.

If d is odd, then there is a linked pair of images of boundaries of $(d+1)/2$-simplices with the vertices at these points. [1]

(VKF_d^+) *'Quantitative' van Kampen-Flores Theorem.* Let $f:\Delta_{d+2}\to\mathbb{R}^d$ be a general position PL map.

If d is even, then the number of intersection points in $\mathbb{R}^d$ of images of disjoint $(d/2)$-faces, is odd. I.e. the number of points $x\in\mathbb{R}^d$ such that $x\in f(\sigma)\cap f(\tau)$ for some disjoint $(d/2)$-faces σ,τ, is odd.

If d is odd, then the number of linked modulo 2 unordered pairs of images of boundaries of $(d+1)/2$-faces with the vertices at these points, is odd. [2]

(TR_d) *Topological Radon Theorem.* For each (continuous or PL) map $\Delta_{d+1}\to\mathbb{R}^d$ there are disjoint faces whose images intersect.

(TR_d^+) *'Quantitative' Topological Radon Theorem.* For each general position PL map $f\colon\Delta_{d+1}\to\mathbb{R}^d$ the number of intersection points in $\mathbb{R}^d$ of images of disjoint k and

I am grateful to S. Avvakumov, R. Karasev, E. Kolpakov, S. Melikhov and M. Skopenkov for useful discussions. Supported in part by RFBR, Grants No. 15-01-06302, the D. Zimin Dynasty foundation, and the Simons-IUM fellowship. Homepage: `www.mccme.ru/~skopenko`. .

[1] This result is due to Conway-Gordon-Sachs for $d=3$ and to Lovas-Schrijver-Taniyama for the general case [CG, Sa81, LS, Ta]. Also there are disjoint $(d-1)/2$ and $(d+1)/2$-faces whose images intersect. This is weaker than (VKF_d) and easily follows from (VKF_{d-1}) by link construction. .

[2] Also the number of intersection points in $\mathbb{R}^d$ of images of disjoint $(d-1)/2$ and $(d+1)/2$-faces, is even (and non-zero, see footnote 1).

$(d-k)$-faces (for all $k = 0, \dots, d$), is odd. I.e. the number of points $x \in \mathbb{R}^d$ such that $x \in f(\sigma) \cap f(\tau)$ for some $k = 0, \dots, d$ and some disjoint k and $(d-k)$-faces σ, τ, is odd.

These well-known results have a vast amount of generalizations, whose citation is outside purposes of this note.

Remark 1 (Main)**.** *(a) There are direct proofs of the following implications (the implications are correct because all the assertions are true).*

$$\begin{tikzcd} VKF_d \arrow[r, bend right, Rightarrow, "(3)"'] & VKF_{d-1} \arrow[l, Rightarrow, "(3)\ d\ even"'] & TR_d \arrow[l, Rightarrow, "(2)\ d\ odd"'] \arrow[r, Rightarrow, "(4)"] & TR_{d-1} \end{tikzcd} \quad \textit{and}$$

$$VKF^+_{d-1} \underset{(3^+)}{\Longleftrightarrow} VKF^+_d \underset{(1^+)}{\Longleftrightarrow} TR^+_d \underset{(4^+)}{\Longrightarrow} TR^+_{d-1} .$$

(b) Direct proof of the right-arrow of (1^+) *is obtained by extending given general position PL map $\Delta_{d+1} \to \mathbb{R}^d$ to a general position PL map $\Delta_{d+2} \to \mathbb{R}^d$ and applying Lemma 2 below (this idea for $d = 2$ appeared in discussions with E. Kolpakov).*

(c) Direct proof of the left-arrow of (1^+) *is obtained by restricting given general position PL map $\Delta_{d+2} \to \mathbb{R}^d$ to Δ_{d+1} and applying Lemma 2 below (this idea appeared in discussions with S. Avvakumov).*

(d) Direct proof of (2) *is a particular case of the Gromov-Blagojević-Frick-Ziegler Constraint Lemma* [Sk16, Lemma 3.2] *for d even. For d odd one can analogously deduce from* (TR_d) *the weaker property of footnote 1.*

(e) Direct proof of (3) *for $d = 2, 4$ was obtained in* [Sk03, Examples 1 and 2], [Ho]. *Direct proof of* (3^+) *for $d = 3$ was obtained in* [RST, Ho, Zi]*; the linear version was discovered by A. Shapovalov around 2003. For a survey see* [Sk14]*. Direct proofs of* (3) *and* (3^+) *for the general case is analogous. For a different proof of a more general statement see* [Me11, Theorem 6.5]*, cf.* [Me06, Example 4.7].[3]

(f) Direct proofs of (3) *and of* (4) *are obtained by cone/link construction.*

(g) Direct proofs of (3^+) *and of* (4^+) *are obtained using Lemma 3 below.*

Cf. [BM] for 'Radon-Tverberg style' proof of the linear versions of (VKF_d).

Now we state and prove Lemmas 2 and 3.

Assume that K is a finite k-complex and $f\colon |K| \to \mathbb{R}^d$ a general position PL map.

Define *the van Kampen number* $v(f) \in \mathbb{Z}_2$ to be the parity of

$$|\{x \in \mathbb{R}^d \ : \ x \in f(\sigma) \cap f(\tau) \text{ for some } \sigma, \tau \in K,\ \sigma \cap \tau = \emptyset,\ \dim\sigma + \dim\tau = d\}|.$$

This is the number of points $x \in \mathbb{R}^d$ such that $x \in f(\sigma) \cap f(\tau)$ for some disjoint simplices $\sigma, \tau \in K$ with $\dim\sigma + \dim\tau = d$. (For another recent application see [ST].)

Define *the Conway-Gordon-Sachs number* $c(f) \in \mathbb{Z}_2$ to be the number modulo 2 of linked unordered pairs of images of $[(d+1)/2]$- and $(d+1-[(d+1)/2])$-faces (or, formally, of their boundaries).

Denote by Δ_N^k the k-skeleton of the N-dimensional simplex. Then

(VKF^+_d) for d even states that $v(f) = 1$ for each general position PL map $f\colon \Delta_{d+2}^{d/2} \to \mathbb{R}^d$,

[3] S. Melikhov kindly informed me of the following. The only if part of [Me06, Theorem 4.2.b] is false as stated. Its special case that is applied in the proof of [Me06, Example 4.7] is correct and is proved by the original arguments. The statement and proof of the general case of [Me06, Theorem 4.2.b] are corrected in [Me11, Theorem 4.6].

(VKF_d^+) for d odd states that $c(f)=1$ for each general position PL map $f\colon \Delta_{d+2}\to\mathbb{R}^d$,

(TR_d^+) states that $v(f)=1$ for each general position PL map $f\colon \Delta_{d+1}\to\mathbb{R}^d$.

Lemma 2. *If $f\colon \Delta_{d+2}\to\mathbb{R}^d$ is a general position PL map, then*

$$v(f|_{\Delta_{d+1}})=\begin{cases} v(f|_{\Delta_{d+2}^{d/2}}) & d \text{ is even}\\ c(f) & d \text{ is odd}\end{cases}.$$

Proof. (A reader can first consider the cases $d=1,2,3$.) For a face σ of Δ_{d+1} denote by $\overline{\sigma}$ the 'complementary' face. Then $\dim\sigma+\dim\overline{\sigma}=d$. Denote by $*$ the vertex of $\Delta_{d+2}-\Delta_{d+1}$. For subcomplexes $A,B\subset\Delta_{d+2}$ the sum of whose dimensions is d denote

$$A\wedge B:=\rho_2|f(A)\cap f(B)|\in\mathbb{Z}_2.$$

Denote by $*A\subset\Delta_{d+2}$ the cone over $A\subset\Delta_{d+1}$ with the vertex $*$. We omit parenthesis assuming that $+$ is the last-to-do operation and $\wedge$ is the last but one operation. Below the summation is over (non-negative-dimensional) faces of Δ_{d+1}, or their pairs, satisfying the assumptions shown under the summation sign. We have

$$v(f|_{\Delta_{d+2}^{d/2}})=\sum_{\substack{\sigma\ :\\ \dim\sigma=d/2-1}} *\sigma\wedge\partial\overline{\sigma}\quad\text{for } d \text{ even}\quad\text{and}$$

$$c(f)=\sum_{\substack{\sigma\ :\\ \dim\sigma=(d-1)/2}}\mathrm{lk}_2(f(\partial(*\sigma)),f(\partial\overline{\sigma}))=\sum_{\substack{\sigma\ :\\ \dim\sigma=(d-1)/2}} *\sigma\wedge\partial\overline{\sigma}\quad\text{for } d \text{ odd}.$$

Now the lemma holds because

$$\left.\begin{array}{ll} d \text{ even}: & v(f|_{\Delta_{d+1}})-v(f|_{\Delta_{d+2}^{d/2}})\\ d \text{ odd}: & v(f|_{\Delta_{d+1}})-c(f)\end{array}\right\}=\sum_{\substack{\sigma\ :\\ \dim\sigma\le[(d-1)/2]}}\sigma\wedge\overline{\sigma}+\sum_{\substack{\sigma\ :\\ \dim\sigma=[(d-1)/2]}} *\sigma\wedge\partial\overline{\sigma}=\ldots=0.$$

Here the first equality is obvious, and the other are obtained by applying the following equality for $k=[(d-1)/2],\ldots,2,1,0$:

$$\sum_{\substack{\sigma\ :\\ \dim\sigma=k}}(\sigma\wedge\overline{\sigma}+*\sigma\wedge\partial\overline{\sigma})\overset{(1)}{=}\sum_{\substack{\sigma\ :\\ \dim\sigma=k}} *\partial\sigma\wedge\overline{\sigma}=\sum_{\substack{(\sigma,\tau)\ :\\ \dim\sigma=k\\ \tau\subset\partial\sigma}} *\tau\wedge\overline{\sigma}\overset{(3)}{=}\sum_{\substack{(\sigma,\tau)\ :\\ \dim\tau=k-1\\ \overline{\sigma}\subset\partial\overline{\tau}}} *\tau\wedge\overline{\sigma}=\sum_{\substack{\tau\ :\\ \dim\tau=k-1}} *\tau\wedge\partial\overline{\tau}.$$

Here[4]

- equality (3) holds because $\tau\subset\partial\sigma\ \Leftrightarrow\ \overline{\sigma}\subset\partial\overline{\tau}$.
- equality (1) follows because by general position $f(*\sigma)\cap f(\overline{\sigma})$ is a finite number of non-degenerate arcs, so

$$*\sigma\wedge\partial\overline{\sigma}=\partial(*\sigma)\wedge\overline{\sigma}+\rho_2|\partial(f(*\sigma)\cap f(\overline{\sigma}))|=\partial(*\sigma)\wedge\overline{\sigma}=\sigma\wedge\overline{\sigma}+*\partial\sigma\wedge\overline{\sigma}.$$

- other equalities are obvious. $\square$

Lemma 3. *Let $f\colon \Delta_{d+2}\to\mathbb{R}^d$ be a general position PL map.*

[4]For $k=0$ the last two sums run over empty set and hence are zeroes. The summands can be non-zero for the (-1)-dimensional face $\tau=\emptyset$ of Δ_{d+1}, but this face is not included into the summation.

(a) Let $\widehat{f}\colon \Delta_{d+3} \to \mathbb{R}^{d+1}$ *be a general position shift of the cone over* f*. Then*

$$\begin{cases} v(\widehat{f}) = v(f) & \\ c(\widehat{f}) = v(f|_{\Delta_{d+2}^{d/2}}) & d \text{ is even} \\ v(\widehat{f}\,|_{\Delta_{d+3}^{(d+1)/2}}) = c(f) & d \text{ is odd} \end{cases}.$$

$$(b) \qquad \begin{cases} c(f) = v((\mathrm{lk}_A f)|_{\Delta_{d+1}^{(d+1)/2}}) & d \text{ is odd} \\ v(f|_{\Delta_{d+2}^{d/2}}) = \sum_A c(\mathrm{lk}_A f) & d \text{ is even} \end{cases}.$$

Here A *is any vertex of* Δ_{d+2}, $\mathrm{lk}_A f : \Delta_{d+1} \to S^{d-1}$ *is any 'link of* f *at* A*', and in the third line the summation is over vertices* A *of* Δ_{d+2}*.*

Proof of Lemma 3 is analogous to the particular cases $d \le 4$ implicitly proved in references for (3) and (3^+) presented in Remark 1.e.

Problem 4. *(a) Find a direct proof of* $TR_d^+ \Rightarrow VKF_{d-1}^+$ *at least for* d *odd, and of* (2) *for* d *even.*

(b) Find direct proofs of the converse implications to (4) *and to* (4^+)*.*

(c) Consider the following assertions for each integers $d, k, r > 0$*:*

($VKF_{k,r}$) (r-fold van Kampen-Flores statement) For each general position PL map $\Delta_{(kr+2)(r-1)}^{k(r-1)} \to \mathbb{R}^{kr}$ *there are* r *pairwise disjoint* $k(r-1)$*-faces whose images have a common point.*

($TT_{d,r}$) (Topological Tverberg statement) For each PL map $\Delta_{(d+1)(r-1)} \to \mathbb{R}^d$ *there are* r *pairwise disjoint faces whose images have a common point.*

Recall that ($TT_{d,r}$) and ($VKF_{k,r}$) are true for r *a prime power and are false otherwise* [Sa91], [Vo, Corollary in §1], [AMSW, Theorem 1.2.a]*; for ($TT_{d,r}$) see surveys* [BZ, Sk16] *and references therein.*

It would be interesting to obtain their 'quantitative' versions. E.g. are the following statements true for primes or prime powers r*?*

($VKF_{k,r}^+$) For each general position PL map $\Delta_{(kr+2)(r-1)}^{k(r-1)} \to \mathbb{R}^{kr}$ *the sum of* r*-intersection signs of intersection points in* $\mathbb{R}^d$ *of images of* r *pairwise disjoint* $k(r-1)$*-faces, is not divisible by* r*.*

($TT_{d,r}^+$) For each (continuous or PL) map $\Delta_{(d+1)(r-1)} \to \mathbb{R}^d$ *the sum of* r*-intersection signs of intersection points in* $\mathbb{R}^d$ *of images of* r *pairwise disjoint faces whose sum of dimensions is* $d(r-1)$*, is not divisible by* r*.*

It would be interesting to find a direct proof of $VKF_{k,r}^+ \Rightarrow TT_{kr,r}^+$ *(can Lemma 2 be generalized?). The implication is correct unless possibly for* $k = 1$ *and* r *a prime power, because both assertions are true or false simultaneously.*

(d) It would be interesting to obtain analogous 'quantitative' version for non-realizability of $K_5 \times K_3$ *in* $\mathbb{R}^3$ *and of* $K_5 \times K_5$ *in* $\mathbb{R}^4$*, cf.* [Sk03, Sk14].

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